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Centralizers of locally nilpotent derivations

David R. Finston [a,*], Sebastian Walcher [b]

[a] *Department of Mathematical Sciences, New Mexico, State University, Las Cruces, NM 88003, USA*

[b] *Mathematisches Institut, Technische Universität München, 80290 München, Germany*

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Abstract

Locally nilpotent derivations of the polynomial ring in n variables over the complex field, algebraic actions of the additive group G_a of complex numbers on $\mathbf{C}^n$, and vector fields on $\mathbf{C}^n$ admitting a strictly polynomial flow, are equivalent objects. The polynomial centralizer of the vector field corresponding to a triangulable locally nilpotent derivation is investigated, yielding a triangulability criterion. Several new examples of nontriangulable G_a actions on $\mathbf{C}^n$ are presented.

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1. Introduction

Every polynomial derivation $\delta = \sum_{i=1}^n p_i \partial/\partial x_i$, $p_i \in \mathbf{C}[x_1, \dots, x_n] = \mathbf{C}[X]$, corresponds to the autonomous polynomial differential equation (vector field P) $\dot{x} = P(x)$ in $\mathbf{C}^n$, where $x = x(t)$ has its values in $\mathbf{C}^n$, and $P(x) = (p_1, \dots, p_n)$. We will not distinguish between the notions of "differential equation" and "vector field" and use the terms synonymously. The derivation is locally nilpotent provided for each $Q \in \mathbf{C}[X]$, some power of δ, depending on Q, annihilates Q. This condition holds if and only if the differential equation has a strictly polynomial flow (i.e. admits a general solution which is polynomial in t). Moreover, in this situation, the assignment $\sigma_t(Q) = \exp(t\delta)(Q)$ gives an algebraic action of the additive group of complex numbers as automorphisms of $\mathbf{C}[X]$ and dually of $\mathbf{C}^n$ [7].

The requirement that the general solution be polynomial in t distinguishes strictly polynomial flow vector fields from polynomial flow vector fields as investigated, for

* Corresponding author.

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instance, in [2]. It is proved there that polynomial flow vector fields correspond to locally finite derivations.

One consequence of the Jung–van der Kulk theorem (see for example [1]) is that all actions of G_a on $\mathbf{C}[x_1,x_2]$ are triangulable. Triangulability of G_a action on $\mathbf{C}[X]$ means that there is a coordinate system $\{u_1,\ldots,u_n\}$ with respect to which the group action has the form $\sigma_t(u_1)=u_1$, $\sigma_t(u_i)=u_i+Q_i$, with $Q_i\in\mathbf{C}[u_1,\ldots,u_{i-1}]$ for $i>1$. The corresponding conditions on the derivation and vector field are easily worked out.

A central question about the structure of the group of polynomial automorphisms of $\mathbf{C}^n$ is whether this group is generated by the triangular and linear automorphisms. The answer to this question is unknown except for the case $n=2$, and several authors have approached this problem by investigating the degree to which embeddings of G_a in this group are triangulable. Bass [1], Popov [9], and Daigle–Freudenburg [4] have given examples of nontriangulable G_a actions on $\mathbf{C}^3$. The Bass and Popov examples all have the form $\sigma_t=\exp(tQ\delta)$, where δ is a (necessarily triangulable) derivation of $\mathbf{C}[x_1,\ldots,x_n]$ defined by a nilpotent linear endomorphism of the complex vector space spanned by $\{x_1,\ldots,x_n\}$ and Q is a suitably chosen element of the kernel of the derivation. In these cases nontriangulability is demonstrated by analyzing the singularities of the fixed point set of the group action. The Daigle and Freudenburg examples are substantially different in that they have nonsingular fixed point sets.

The methods presented here utilize the Lie algebra structure on the vector space of derivations of $\mathbf{C}[x_1,\ldots,x_n]$ (equivalently the Lie algebra of polynomial vector fields on $\mathbf{C}^n$). In particular we consider the centralizer of a given vector field, a structure of central importance in S. Lie's fundamental work on differential equations. As will be seen, this structure enables one to demonstrate nontriangulability of a large class of polynomial vector fields in a completely clear and elementary way.

2. Generalities on vector fields and centralizers

Let δ be a derivation of $\mathbf{C}[X]$ and P the corresponding vector field. The Lie derivative of a rational function $\phi\in\mathbf{C}(X)$ with respect to P, written $L_P(\phi)$, is equal to $\delta(\phi)$, where δ has been extended to rational functions in the obvious way. A semi-invariant for P is a function μ satisfying $L_P(\mu)=\psi\mu$ for some polynomial ψ, and a first integral of P is a constant (i.e. element of the kernel) of δ. The following assertions are mostly well known, and are straightforward consequences of local nilpotency and unique factorization in $\mathbf{C}[X]$ (see [10] for example).

Remark 2.1. (1) If $\dot{x}=P(x)$ with $P(x)=(p_1,\ldots,p_n)$ has a strictly polynomial flow, then the stationary points of the flow are exactly the common zeros of the polynomials $\{p_1,\ldots,p_n\}$.

(2) Every semi-invariant of a strictly polynomial flow vector field is a first integral. (If $L_P(\mu)=\psi\mu$, the associated action of the additive group as automorphisms of $\mathbf{C}[X]$ will satisfy $\sigma_t(\mu)=\mu\rho(X,t)$, for some polynomial ρ. Unique factorization in $\mathbf{C}[X]$

shows that $\rho(X,\alpha) \in \mathbf{C}$ for all complex numbers α. Thus $\rho \in \mathbf{C}$, and the assignment $t \mapsto \rho$ defines a character of the additive group, forcing $\rho = 1$. (I.e. $L_P(\mu) = \psi\mu$ forces $L_P(\mu) = 0$.)

(3) Every rational first integral of P is a ratio of polynomial first integrals. (Apply the quotient rule and assertion 2.)

(4) Let $\delta = \sum_{i=1}^{n} g_i \partial/\partial x_i$ and $\mu \in \mathbf{C}[X]$. Then $\mu\delta$ is locally nilpotent if and only if δ is locally nilpotent and μ is in the kernel of δ. (I.e. μG admits a strictly polynomial flow iff G does and μ is a first integral of G.)

Locally nilpotent derivations δ as above with $\gcd\{g_i\} = 1$ have been termed primitive (e.g. [4]). As in the remark, any locally nilpotent derivation is a multiple of a primitive one by a first integral, and we will refer to differential equations and vector fields as primitive analogously.

A differential equation $\dot{x} = P(x)$ (equivalently derivation $\sum_{i=1}^{n} p_i \partial/\partial x_i$) is said to be triangular provided $p_1 = 0$ and for $i > 1$, $p_i \in \mathbf{C}[x_1, \dots, x_{i-1}]$. It is said to be triangulable if there is a polynomial automorphism of $\mathbf{C}^n$ which transforms the differential equation to a triangular one, and rationally triangulable if there is a birational transformation of $\mathbf{C}^n$ for which the transformed derivation has the form $\sum_{i=1}^{n} q_i \partial/\partial x_i$ with $q_1 = 0$, and for $i > 1$, $q_i \in \mathbf{C}(x_1, \dots, x_{i-1})$. (See [6] for results on rational triangulability.)

For any $F \in \mathbf{C}(X)^n$ denote the $n \times n$ Jacobian matrix of F by DF. If $G = (g_1, \dots, g_n)^{\mathrm{T}}$ is any rational map, $DF \cdot G$ denotes the rational map $x \to DF(x) \cdot G(x)$, and $[G,F] \equiv DF \cdot G - DG \cdot F$.

Note that any derivation of $\mathbf{C}[X]$ (resp. $\mathbf{C}(X)$) can be written uniquely as $\sum_{i=1}^{n} f_i \partial/\partial x_i$ with $f_i \in \mathbf{C}[X]$ (resp. $f_i \in \mathbf{C}(X)$). Let $F = (f_1, \dots, f_n)^{\mathrm{T}}$, and denote this derivation by δ_F. Given derivations δ_F and δ_G their commutator $[\delta_G, \delta_F]$ is again a derivation, hence of the form δ_H. Once checks easily that $H = [G,F]$.

Definition 2.2. For a derivation δ of $\mathbf{C}[X]$, its polynomial (resp. rational) centralizer $\mathscr{C}_{\mathrm{pol}}(\delta)$ (resp. $\mathscr{C}_{\mathrm{rat}}(\delta)$) is the collection of derivations ρ of $\mathbf{C}[X]$ (resp. $\mathbf{C}(X)$) for which $[\rho, \delta] = 0$.

In terms of vector fields, the definition becomes

Definition 2.2′. For a polynomial vector field $\dot{x} = F(x)$, its polynomial (resp. rational) centralizer $\mathscr{C}_{\mathrm{pol}}(F)$ (resp. $\mathscr{C}_{\mathrm{rat}}(F)$) is the collection of polynomial (resp. rational) mappings G for which $[G,F] = 0$.

Since each δ is of the form δ_F for some F, there is no ambiguity in identifying $\mathscr{C}_{\mathrm{pol}}(\delta)$ with $\mathscr{C}_{\mathrm{pol}}(F)$, similarly for the rational centralizers.

The following observations about polynomial and rational centralizers of a locally nilpotent derivation δ_F (strictly polynomial flow vector field $\dot{x} = F(x)$) are common knowledge and easily verified:

1. $\mathscr{C}_{\mathrm{rat}}(F)$ and $\mathscr{C}_{\mathrm{pol}}(F)$ are Lie algebras over $\mathbf{C}$.

2. $\mathscr{C}_{\mathrm{pol}}(F)$ is a module over the kernel of δ_F (= ring $\mathbf{C}[X]^{G_a}$ of invariants of the associated G_a action on $\mathbf{C}[X]$ = ring of polynomial first integrals of the vector field).
3. $\mathscr{C}_{\mathrm{rat}}(F)$ is a vector space over the field of G_a invariants (= field of rational first integrals).
4. $\mathscr{C}_{\mathrm{pol}}(F)$ is not a Lie algebra over $\mathbf{C}[\mathbf{X}]^{G_a}$, since $[\mu G, F] = \delta_G(\mu)F + \mu[G,F] = \delta_G(\mu)F$ $(= L_G(\mu))$.

Lemma 2.3. *Let $\mu, g_1, \ldots, g_n \in \mathbf{C}[x_1, \ldots, x_n]$ be relatively prime, and $\frac{1}{\mu}G : \mathbf{C}^n \to \mathbf{C}^n$ the so determined rational function. Then $\frac{1}{\mu}G$ is in the centralizer of the differential equation $\dot{x} = F(x)$ with strictly polynomial flow if and only if $G \in \mathscr{C}_{\mathrm{pol}}(F)$ and μ is a first integral of $\dot{x} = F(x)$.*

Proof. Assume that $\frac{1}{\mu}G$ is in the centralizer. Then

$$0 = [F, \tfrac{1}{\mu}G] = \tfrac{-1}{\mu^2}L_F(\mu)G + \tfrac{1}{\mu}[F, G]$$

yields $\mu[F,G] = L_F(\mu)G$. But μ and g_i relatively prime shows that $L_F(\mu)$ is a multiple of μ, forcing $L_F(\mu) = 0$.

The other direction is obvious. □

A similar argument proves

Lemma 2.4. *Let $\mu, g_1, \ldots, g_n \in \mathbf{C}[x_1, \ldots, x_n]$ with the g_i relatively prime and $\dot{x} = F(x)$ a differential equation with strictly polynomial flow. Then $\mu G \in \mathscr{C}_{\mathrm{pol}}(F)$ if and only if $G \in \mathscr{C}_{\mathrm{pol}}(F)$ and $L_F(\mu) = 0$.*

The next lemma concerns the behavior of rational centralizers under changes of variables. Note that if $\Phi : \mathbf{C}^n \to \mathbf{C}^n$ is birational, then the change of variables $y = \Phi(x)$ transforms the differential equation $\dot{x} = F(x)$ to $\dot{y} = F^*(y)$, where $F^*(\Phi(x)) = D\Phi(x)F(x)$. In terms of derivations, $\delta_F = [D\Phi(x)]^{-1}\delta_{F^* \circ \Phi}$. For a rational map $G : \mathbf{C}^n \to \mathbf{C}^n$ define $G^{\Phi}(x) = [D\Phi(x)]^{-1}G(\Phi(x))$. With these notations, the following lemma is clear.

Lemma 2.5. *The assignment $G \mapsto G^{\Phi}$ defines an isomorphism from $\mathscr{C}_{\mathrm{rat}}(F^*)$ to $\mathscr{C}_{\mathrm{rat}}(F)$. If Φ is a polynomial automorphism of $\mathbf{C}^n$, then we obtain an isomorphism from $\mathscr{C}_{\mathrm{pol}}(F^*)$ to $\mathscr{C}_{\mathrm{pol}}(F)$.*

The last lemma of this section, although nearly obvious, provides the crucial triangulability criterion.

Lemma 2.6. *Let $\dot{x} = F(x)$ be triangulable. Then there exists a $G \in \mathscr{C}_{\mathrm{pol}}(F)$ satisfying*
(1) G has strictly polynomial flow and
(2) G has no stationary points.

Proof. By the previous lemma we may assume that $\dot{x} = F(x)$ is triangular, in which case $G = (0, \ldots, 0, 1)$ satisfies the assertion. □

Together with a classical result on the structure of the automorphism group of a polynomial ring in two variables over the complex field, the lemma provides a necessary and sufficient condition for triangulability.

Theorem 2.7. *A locally nilpotent derivation δ_F of $\mathbf{C}[x_1,x_2,x_3]$ (strictly polynomial flow vector field $\dot{x} = F(x)$ on $\mathbf{C}^3$) is triangulable if and only if $\mathscr{C}_{\mathrm{pol}}(F)$ contains a locally nilpotent derivation (polynomial vector field) for which the associated G_{a} action is conjugate to a translation (whose flow can be straightened polynomially).*

Proof. One direction is provided by Lemma 2.6. For the other direction, assume that $(0,0,1) \in \mathscr{C}_{\mathrm{pol}}(F)$. A simple computation reveals that $F(x) = (f_1, f_2, f_3)$ where all $f_i \in \mathbf{C}[x_1,x_2]$. Since $\dot{x} = F(x)$ has strictly polynomial flow, so does the vector field (f_1, f_2) defined on $\mathbf{C}^2$. As a consequence of the Jung–van der Kulk theorem [1], the latter vector field is triangulable. □

Remark 2.8. Triangulability of a locally nilpotent derivation is equivalent to the triangulability of the G_{a} action it generates, i.e. simultaneous triangulability of all σ_t, $t \in \mathbf{C}$. Moreover, if σ_a is triangulable for some $a \neq 0$, then σ_t is triangulable for all t. Indeed, $\{b \in \mathbf{C}\colon \sigma_b \text{ is triangulable}\}$ is an algebraic subgroup of G_{a}, hence the trivial subgroup or all of G_{a}.

3. Popov's criterion

Popov's triangulability criterion [9] states that if a strictly polynomial flow vector field is triangulable, then its set of stationary points is a cylindrical variety, i.e. isomorphic to the product of some affine variety with a line. In all the examples of nontriangulable vector fields in that paper, the failure of the varieties of stationary points to be cylindrical is due to the presence of isolated singularities. A generalization of this aspect of the criterion is given by

Proposition 3.1. *Let $\dot{x} = F(x)$ have strictly polynomial flow, $F \neq 0$, and μ a polynomial first integral of $\dot{x} = F(x)$ for which $D\mu$ vanishes at an isolated point. Then $\dot{x} = \mu(x)F(x)$ is not triangulable.*

Proof. Assume that $\dot{x} = \mu(x)F(x)$ is triangulable, and G is as in Lemma 2.6. From $[G, \mu F] = 0$ we obtain $L_G(\mu) = 0$. The set of singular points of μ (i.e. those points where the Jacobian vanishes) is invariant for the differential equation $\dot{x} = G(x)$ [12, Proposition 3.11]. But then an isolated singular point of μ is a stationary point for $\dot{x} = G(x)$, a contradiction.

Examples

1. (Bass–Nagata [1]) $\dot{x} = (x_2^2 - 2x_1x_3)(0,x_1,x_2)$. The point $(0,0,0)$ is an isolated singularity of $(x_2^2 - 2x_1x_3)$.
2. $\dot{x} = (x_2^2 - 2x_1x_3 + 1)(0,x_1,x_2)$. The argument of [9] does not apply here, because the variety of stationary points is nonsingular. However, the proposition again applies with the point $(0,0,0)$ to show nontriangulability.
3. $\dot{x} = (x_1x_4 - x_2x_3 + 1)(0,x_1,0,x_3)$. Again nonsingularity of the variety of stationary points precludes the use of Popov's criterion, but the isolated singular point of $(x_1x_4 - x_2x_3 + 1)$ at the origin and the proposition imply nontriangulability.
4. $\dot{x} = [1+(x_1x_4-x_2x_3)^2](0,x_1,0,x_3)$. Here the set of singular points of $1+(x_1x_4-x_2x_3)^2$ is the hypersurface defined by $(x_1x_4-x_2x_3)$; as such it has no isolated singular points. However, the variety of singular points is an invariant set for the differential equation and has an isolated singularity at the origin. Since this point is stationary for any centralizer element by [12], Lemma 2.6 yields nontriangulability.
5. $\dot{x} = ((x_2^2 - 2x_1x_3)x_1 + 1)(0,x_1,x_2)$. Here the fixed point set consists of the (disjoint) union of the line $x_1 = x_2 = 0$ and the smooth surface $(x_2^2 - 2x_1x_3)x_1 + 1 = 0$. Thus Popov's method does not apply. Neither does the method of Proposition 3.1, since $D((x_2^2-2x_1x_3)x_1+1)$ vanishes only along the line $x_1 = x_2 = 0$. However, the methods developed in the next section show that this example is also nontriangulable.

4. Popov actions

The locally nilpotent derivations corresponding to the examples closing the previous section have the form $\delta = f\hat{\delta}$, where the derivation $\hat{\delta}$ restricts to a nilpotent linear mapping of the $\mathbf{C}$ linear span of $\{x_1,\dots,x_n\}$, and f is an invariant of the associated G_a action (i.e. a first integral of the associated differential equation). The G_a actions generated by such derivations were termed Popov actions in [5], and shown there to be rationally triangulable. Most of these actions, however, are not triangulable.

In this section consider a triangular equation $\dot{x} = (0, p(x_1), q(x_1,x_2)) = F(x)$ with the assumptions

1. p and q are relatively prime, and
2. $p(0) = q(0,0) = 0$.

It is well known that the ring of polynomial first integrals is generated by $\phi_1 = x_1$ and $\phi_2 = x_3p(x_1) - h(x_1,x_2)$, where $h = \int_0^{x_2} q(x_1,u)\,\mathrm{d}u$. Set $\phi_3 = x_2/p$, and $\Phi = (\phi_1,\phi_2,\phi_3)$. Note that Φ is a birational mapping of $\mathbf{C}^3$. We use the result of Lemma 2.5 to calculate the polynomial centralizer of F.

Observe that

$$D\Phi(x) = \begin{pmatrix} 1 & 0 & 0 \\ p'x_3 - \dfrac{\partial h}{\partial x_1} & -q & p \\ -\dfrac{p'x_2}{p^2} & \dfrac{1}{p} & 0 \end{pmatrix},$$

and that

$$D\Phi(x)F(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

With T equal to the mapping $\mathbf{C}^3 \to \mathbf{C}^3$ given by $(0,0,1)$, Lemma 2.5 shows that the mapping $F \mapsto D\Phi(x)^{-1}F(\Phi(x))$ defines an isomorphism from $\mathscr{C}_{\mathrm{rat}}(T) = \{(g_1,g_2,g_3): g_i \in \mathbf{C}(x_1,x_2)\}$ to $\mathscr{C}_{\mathrm{rat}}(F)$.

A computation reveals that

$$D\Phi(x)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{p'x_2}{p} & 0 & p \\ * & \dfrac{1}{p} & q \end{pmatrix}.$$

Thus every element of the polynomial centralizer of F has the form

$$(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3) = \left(g_1(x_1,\phi_2), \frac{p'(x_1)}{p(x_1)}x_2g_1(x_1,\phi_2) + p(x_1)g_3(x_1,\phi_2), * \right). \tag{$*$}$$

Here the g_i may a priori be rational functions.

Lemma 4.1. (1) *If $\tilde{G} = (\tilde{g}_1,\tilde{g}_2,\tilde{g}_3)$ lies in $\mathscr{C}_{\mathrm{pol}}(F)$, then x_1 divides $\tilde{g}_1$.*
(2) *If $\tilde{G}$ has strictly polynomial flow, then $\tilde{g}_1 = 0$.*

Proof. It suffices to prove the first assertion, since x_1 will then have been shown to be a semi-invariant, hence a first integral of $\tilde{G}$.

From $(*)$ we know that $\tilde{g}_1 = g_1(x_1,\phi_2)$ and g_1 is therefore a polynomial. Consequently, we deduce from $(*)$ that $g_3 = r/s$ where $r = r(x_1,\phi_2)$ and $s = s(x_1)$ are relatively prime polynomials. Write $p(x_1) = x_1^n\tilde{p}(x_1)$, where $n \geq 1$ by hypothesis. Again $(*)$ shows that

$$p'(x_1)x_2sg_1 + p^2(x_1)r = p(x_1)s\tilde{g}_2. \tag{$**$}$$

Thus s divides $p^2(x_1)$. Write $s = x_1^m\tilde{s}$ with $\tilde{s}(0) \neq 0$.

Assume that x_1 does not divide $\tilde{g}_1$. Then the highest powers of x_1 dividing $p'x_2sg_1$ and p^2r are x_1^{n+m-1}, x_1^{2n}, respectively, while x_1^{n+m} divides $ps\tilde{g}_2$. These conditions and $(**)$ show that $m = n+1$ and therefore $(**)$ reduces to

$$(n\tilde{p} + x_1\tilde{p}')\tilde{s}x_2g_1(x_1,\phi_2) + \tilde{p}^2r = x_1\tilde{s}\tilde{p}\tilde{g}_2(x_1,x_2,x_3). \tag{$***$}$$

Recall that $\phi_2 = x_3p(x_1) - h(x_1,x_2)$ where $h = \int_0^{x_2} q(x_1,u)\,\mathrm{d}u$, and that $p(0) = q(0,0) = 0$. Since p and q are relatively prime, $q(0,x_2)$ has positive degree and therefore the degree of $h^* = h(0,x_2)$ is greater than one. Evaluate $(***)$ at $x_1 = x_3 = 0$ to obtain $\alpha x_2g_1(0,h^*) + \beta r(0,h^*) = 0$, with α and β nonzero constants. This places $x_2 \in \mathbf{C}(h^*)$, which is impossible. $\square$

Theorem 4.2. *Let $\dot{x} = F(x)$ be triangular, with a stationary point at the origin, and x_1, ϕ_2 the generators of the ring of polynomial first integrals described above. For any bivariate polynomial $\sigma(u,v)$, the differential equation $\dot{x} = \sigma(x_1,\phi_2)F(x) = \mu(x)F(x)$ is nontriangulable if $\partial\sigma/\partial v \neq 0$.*

Proof. Suppose that $\tilde{G} \in \mathscr{C}_{\text{pol}}(\mu F)$, and that $\tilde{G}$ has strictly polynomial flow. According to Lemma 2.4 then, $[\tilde{G},F] = 0$ and $L_{\tilde{G}}(\mu) = 0$. From Lemma 4.1 we obtain $\tilde{G} = (0,\tilde{g}_2,\tilde{g}_3)$, so that x_1 is a first integral for $\dot{x} = \tilde{G}(x)$. The calculation

$$0 = L_{\tilde{G}}(\mu) = \frac{\partial\sigma}{\partial u}(\phi_1,\phi_2)L_{\tilde{G}}(\phi_1) + \frac{\partial\sigma}{\partial v}(\phi_1,\phi_2)L_{\tilde{G}}(\phi_2)$$

yields $L_{\tilde{G}}(\phi_2) = 0$.

The calculation $\tilde{G}(x) = D\Phi(x)^{-1}G(\Phi(x))$ yields (with $p = p(x_1)$, $g_i = g_i(x_1,\phi_2)$) $\tilde{G} = (0, pg_3, (1/p)g_2 + q(x_1,x_2)g_3)$, which we write as $g_2(0,0,1/p) + g_3F$. Since ϕ_2 is a first integral of $\dot{x} = F(x)$ and $\phi_2 = p(x_1)x_3 - h(x_1,x_2)$ we obtain $0 = L_{\tilde{G}}(\phi_2) = g_2(x_1,\phi_2)$. In particular, the elements of $\mathscr{C}_{\text{pol}}(F)$ which have strictly polynomial flow are all of the form $\rho(x_1,\phi_2)F$ for $\rho \in \mathbf{C}[u,v]$. Since the flow of every such centralizer element has a stationary point (namely zero), Lemma 2.6 shows that $\dot{x} = \mu(x)F(x)$ is not triangulable. □

Using other methods, Daigle obtained essentially the same result in [3]. In the terminology therein, a derivation δ of $\mathbf{C}[X]$ has corank j provided j is the largest integer for which $\mathbf{C}^n$ has a coordinate system $\{u_1,\ldots,u_n\}$ with $\{u_1,\ldots,u_j\}$ in the kernel of δ. The rank of δ is then $n-j$. Daigle's hypothesis is that the triangular derivation F have rank two on $k[X]$, where $X = k^3$ and k is any field of characteristic 0. For $k = \mathbf{C}$ it is known that any rank two triangular vector field on $\mathbf{C}^3$ has a stationary point, since fixed point free triangular G_a actions on complex three space are conjugate to translations [11], and therefore have rank 1. In a very recent preprint, Freudenburg has given an example of a rank three derivation of the polynomial ring in three variables whose associated G_a action has stationary points.

5. Examples

The methods of the previous sections are applied to show that none of the non Popov polynomial vector fields $\dot{x} = (0, x_1(x_2^2 - 2x_1x_3), x_1^n + x_2(x_2^2 - 2x_1x_3))$ are triangulable for $n \geq 0$. The case $n = 1$ was considered by Daigle and Freudenburg [4]. These examples are noteworthy, since their sets of stationary points are finite unions of lines rather than hypersurfaces as in previously known nontriangulable vector fields. For the record, and future use, the ring of polynomial first integrals for this vector field is generated over $\mathbf{C}$ by x_1 and $-x_1^nx_2 + \frac{1}{2}(x_1x_3 - \frac{1}{2}x_2^2)^2$. These are easily shown to be first integrals, and the algorithm in [8] shows that they generate.

Consider the equation $\dot{x} = F^*(x) \equiv (0, x_1^m, x_2)$ with $m \geq 1$. The ring of invariants of the associated G_a action is generated by x_1 and $\phi_2 = x_1^m x_3 - x_2^2/2$.

As a module over $\mathbf{C}[x_1, \phi_2]$, $\mathscr{C}_{\mathrm{pol}}(F^*)$ is generated by $(0,0,1)$, F^*, and (x_1, mx_2, mx_3). To see this, observe that with $\Psi(x)$ denoting the birational mapping $(x_1, \phi_2, x_2/x_1^m)$, we have $D\Psi(x)F^*(x) = (0,0,1) = T(x)$. Thus the mapping $G \mapsto D\Psi(x)^{-1}G(\Psi(x))$ defines an isomorphism from $\mathscr{C}_{\mathrm{rat}}(T)$ to $\mathscr{C}_{\mathrm{rat}}(F^*)$. A basis for the former as a vector space over the field of invariant rational functions is given by $\{(0,1,0),(0,0,1),(1,0,0)+(2\phi_2/x_1)(0,1,0)\}$. Applying the isomorphism, and clearing denominators, yields the desired generating set for the module of polynomial centralizer elements.

Consider the vector field $\dot{x} = F(x) \equiv (0, x_1(x_1x_3 - x_2^2/2), x_1^n + x_2(x_1x_3 - x_2^2/2))$. Observe that $G(x) = (0, x_1, x_2)$ lies in the centralizer of F and that the birational mapping $\Phi(x) = (x_1, x_1x_3 - x_2^2/2,\ x_2/x_1)$ straightens G in the sense that $D\Phi(x)G(x) = (0,0,1)$. Moreover, $D\Phi(x)F(x) = (0, x_1^{n+1}, 0) + (x_1x_3 - x_2^2/2)(0,0,1) \equiv F^*(\Phi(x))$, where $F^*(x) = (0, x_1^{n+1}, x_2)$.

Since Φ is birational, we obtain as above an isomorphism from $\mathscr{C}_{\mathrm{rat}}(F^*)$ to $\mathscr{C}_{\mathrm{rat}}(F)$. Applying this isomorphism to the generating set of polynomial centralizers for F^* found above (with $m = n+1$) we obtain the basis $\{G(x), F(x), K(x) = (1/x_1)(x_1^2, (n+2)x_1x_2, nx_1x_3 + ((3+n)/2)x_2^2)\}$ for $\mathscr{C}_{\mathrm{rat}}(F)$ as a vector space over the associated field of invariant rational functions. In particular, every polynomial centralizer element H of F can be expressed as

$$\begin{aligned} H &= \alpha(x_1, \psi_2)(x_1^2, (n+2)x_1x_2, nx_1x_3 + ((3+n)/2)x_2^2) \\ &\quad + \beta(x_1, \psi_2)(0, x_1(x_1x_3 - \tfrac{1}{2}x_2^2), x_1^n + x_2(x_1x_3 - \tfrac{1}{2}x_2^2)) \\ &\quad + \gamma(x_1, \psi_2)(0, x_1, x_2) \end{aligned}$$

with α, β, γ bivariate rational functions, and $\psi_2(x) = -x_1^n x_2 + \frac{1}{2}(x_1x_3 - \frac{1}{2}x_2^2)^2$.

It will be seen that any such H, for which $\dot{x} = H(x)$ has strictly polynomial flow, will necessarily have a stationary point. Lemma 2.6 will then show that F is nontriangulable.

Since H is polynomial, $x_1^2\alpha$ is a polynomial, hence $\alpha = \tilde{\alpha}/x_1^2$ for some polynomial first integral $\tilde{\alpha}$. In fact, we will show that $\tilde{\alpha}$ is divisible by x_1, so that $H = (x_1\tilde{h}_1, h_2, h_3)$. For H to have strictly polynomial flow however, $\tilde{h}_1 = 0$ (e.g. Remark 2.1).

Again using the fact that H is polynomial, we see that $(\tilde{\alpha}/x_1^2)[(n+2)x_1x_2^2 - nx_1^2x_3 - ((n+3)/2)x_1x_2^2] - \beta x_1^{n+1}$ is a polynomial. Clearing denominators and cancelling like powers of x_1 yields $\tilde{\alpha}(x_1, \psi_2)[((n+1)/2)x_2^2 - nx_1x_3] - \tilde{\beta}(x_1, \psi_2) = x_1 p$ for some polynomial p.

Evaluate at $x_1 = 0$, to obtain in

- Case $n > 0$, $\tilde{\alpha}(0, x_2^4/8)(((n+1)/2)x_2^2) - \tilde{\beta}(0, x_2^4/8) = 0$. Unless $\tilde{\alpha}(0,*) = 0$, i.e. unless x_1 divides $\tilde{\alpha}$, we obtain the absurdity that $x_2^2 \in \mathbf{C}(x_2^4)$.
- Case $n = 0$, by a similar argument, that unless $\tilde{\alpha}(0,*) = 0$, we have the absurdity $x_2^2 \in \mathbf{C}(x_2 - x_2^4/8)$.

Thus any element of $\mathscr{C}_{\mathrm{pol}}(F)$ that has strictly polynomial flow, must be of the form $H = \beta(x_1, \phi_2)(0, x_1c, x_1^n + x_2c) + \gamma(x_1, \phi_2)(0, x_1, x_2)$, with $c = x_1x_3 - \frac{1}{2}x_2^2$, and β and γ rational functions.

However, β and γ are in fact polynomials, and this suffices for our purposes. Indeed, assuming this assertion, if $n > 0$, then the line $x_1 = x_2 = 0$ consists of stationary points for the flow determined by H, so that $\dot{x} = F(x)$ is not triangulable. If $n = 0$, evaluate H at $x_1 = 0$ to obtain $H(0, x_2, x_3) = (0, 0, \beta(0, x_2^4/8)(1 - x_2^3/2) + \gamma(0, x_2^4/8)x_2)$. The third coordinate, as a polynomial in x_2, cannot be a non zero constant, since the degree of the first summand is congruent to $3 \bmod 4$, and the second summand has degree congruent to $1 \bmod 4$. For any root r of this polynomial, the line $x_1 = 0$, $x_2 = r$ consists of stationary points for the flow determined by H.

To see that β and γ are polynomials, first note that x_2(second coordinate of H) $-$ x_1(third coordinate of H) $= x_1^{n+1}\beta$ which is a polynomial. Write $\beta = \tilde{\beta}/x_1^{n+1}$ so that the second (polynomial) coordinate in H becomes $(\tilde{\beta}/x_1^n)c + x_1\gamma$. We obtain $c\tilde{\beta} + x_1^{n+1}\gamma = x_1^n Q$ for some polynomial Q. In particular, $x_1^{n+1}\gamma = \tilde{\gamma}$, a polynomial, and

$$c\tilde{\beta} + \tilde{\gamma} = x_1^n Q. \tag{$*$}$$

If $n > 0$, evaluate $(*)$ at $x_1 = 0$ to obtain $-\tilde{\beta}(0, \frac{1}{8}x_2^4)x_2^2 + \tilde{\gamma}(0, \frac{1}{8}x_2^4) = 0$. Unless both $\tilde{\beta}(0, z)$ and $\tilde{\gamma}(0, z)$ are the zero polynomial in z, we obtain the contradiction $x_2^2 \in \mathbf{C}(x_2^4)$. Thus x_1 divides $\tilde{\beta}$ and $x_1^n\beta$ is a polynomial.

The third coordinate of H shows that $x_2(c\beta + \gamma)$ is a polynomial. But the denominators of both β and γ are at worst powers of x_1, so that $c\beta + \gamma$ is a polynomial. Writing $\beta = b/x_1^s$ and $\gamma = g/x_1^t$ with polynomials b and g, neither of which is divisible by x_1, we obtain one of the relations $cb + x_1^i g = x_1^j Q$ or $x_1^i cb + g = x_1^j Q$. Evaluation at $x_1 = 0$ and an argument as in the previous paragraph show that $s = t = 0$.

The case $n = 0$ is handled similarly. Since x_2 (second coordinate of H) $-x_1$ (third coordinate of H) is a polynomial, we obtain $\beta = \tilde{\beta}/x_1$ and $\gamma = \tilde{\gamma}/x_1$. The third coordinate of H shows that $(1 + x_2 c)\tilde{\beta} + x_2\tilde{\gamma} = x_1 Q$ for some polynomial Q, so that if x_1 divides $\tilde{\beta}$ (i.e. β is a polynomial), then γ is a polynomial. Evaluation at $x_1 = x_2 = 0$ yields

$$0 = \tilde{\beta}(0, -x_2 + \tfrac{1}{8}x_2^4)(1 - \tfrac{1}{2}x_2^3) + \tilde{\gamma}(0, -x_2 + \tfrac{1}{8}x_2^4).$$

Unless $\tilde{\beta}(0, z)$ is the zero polynomial, this last equation places $(1 - \frac{1}{2}x_2^3)/x_2$ in the field $\mathbf{C}(-x_2 + \frac{1}{8}x_2^4)$, a contradiction.

It is remarked in [4] that the proof given by Daigle and Freudenburg can be modified to show nontriangulability if n is not a multiple of 3.

It should be noted that whenever $\dot{x} = F(x)$ and $\dot{x} = G(x)$ have strictly polynomial flows, $[F, G] = 0$, and μ is a first integral of G, then $\dot{x} = F(x) + \mu(x)G(x)$ will have strictly polynomial flow. This seems to be the appropriate generalization of the Daigle–Freudenburg examples to arbitrary dimension.

In view of the results in Section 4 the following conjecture for $\mathbf{C}^3$ seems reasonable.

Let $\dot{x} = G(x)$ be triangular with first integrals generated by x_1 and ϕ_2. Let $\sigma(u, v)$ be a polynomial such that $\partial\sigma/\partial v \neq 0$, and let τ be any polynomial in one variable. Then

$$\dot{x} = (0, 0, \tau(x_1)) + \sigma(x_1, \phi_2)G(x)$$

is not triangulable provided that G has stationary points.

The condition on σ is obviously necessary while the condition on G is necessary since otherwise G can be straightened to $(0,0,1)$ by a polynomial automorphism [4, Corollary 3.3] and triangulability follows easily.

In principle, a proof of the above conjecture should be possible using the same strategy as in the Daigle–Freudenburg example, since a birational map transforming the given vector field into a constant one can be obtained. The technical difficulties increase however, and it seems likely that some nonelementary arguments are required.

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